Looking at the Pazzi Chapel's umbrella vault from its oculus

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Abstract: The subject of the research is the umbrella vault in the Pazzi Chapel in Santa Croce church in Florence. In some previous researches done by the author, attempts were made to formulate new hypotheses on the form-finding processes that could have determined the shape of umbrella's vault sail. The analyses were done by correlating detailed surveys and geometric analyses of the vault, comprising the curves, surfaces and the possible masonry texture with the hypothesized form. The hypothesized form of the inner sail is obtained by a three-dimensional transformation of a toric surface, with respect to the fixed point (dome's oculus) and constant length (torus radius), like the conchoid of Nicomedes in 2d. The vault is made by the inner sail (the conchoid surface) and the outer sail that is not visible in this moment (the hypothesized toric surface). The results were verified by overlapping the hypothesized inner sail's form with the laser scanner data, and only 4% of points were out of the 3 cm range. This very particular form, even though it explains the building process in a very satisfactory manner, is not actually something that the visitors can see (with the naked eye). Actually, the conchoid surface (that is a solid central projection of torus), could be perceived like a transformation of torus only under certain conditions.
First condition: the inner conchoidal sail's surface should be mapped by the elements that represent the projection of the torus's peculiar elements (circumferences for example) onto the conchoid surface from the fixed point (oculus). These irregular spatial curves, that represent the regular curve's transformation, can be perceived as if they were circular if we look at them from the transformation center. The second condition would be the transportation of the observer's view point in the center of oculus (not accessible point). The exhibition design (work in progress for the Opera of Santa Croce) involves the use of projectors for the 3d mapping of the sails and use of a plane mirrors system that can move the observer's view point as if he was in the cupola's oculus.

Keywords: 3d Conchoid surface, Torus, Mirror, 3d Mapping, Solid central projection

Introduction

The Pazzi Chapel in Santa Croce church in Florence is considered one of the cornerstones of Renaissance architecture, and to most scholars, it marks one of the highlights of the career of the great architect Filippo Brunelleschi, who is best known as the man who designed the dome of the cathedral in Florence. The chapel, put up in the mid-1400's by the Medici rivals, the Pazzi banking family, seems to sum up his ideas of Renaissance architecture perfectly. The building was started between 1429 and 1430, but the date in which it was completed is not certain and it ranges from 1443 and 1478, in any case after Brunelleschi's death (1446). The structural system is the same like one he adopted in the Old Sacristy of San Lorenzo and Duomo's cupola afterwards, and it's based on a double-sailed vault structure.

According to some scholars it was not Brunelleschi that designed the Pazzi Chapel but Michelozzo that copied it from the Old Sacristy: Marvin Trachtenberg argued that is unlikely that an architect as powerful and influential as Brunelleschi would have allowed an immature design and it is all but unthinkable that so potent and creative force as Brunelleschi would actually have offered up an unrefined copy of his early work (TRACHTENBERG 1996, 1997, 2008). In the Pazzi Chapel, Trachtenberg points out, there is little of the direct and visible connection between ornament and structure that helps make the Old Sacristy so notable; and the ornament, while superficially similar to that of the Old Sacristy, has "a surprising roughness and clumsiness" (TRACHTENBERG 1997). The Trachtenberg's arguments are based on evidences that came through an intense study of the building itself, especially its ornaments and structure.

Fig. 1 – Pazzi Chapel in Santa Croce Church in Florence: external view from the cloister (above) and internal view of umbrella vault (below).

For years, scholars have offered theories about why the chapel was not entirely up to Brunelleschi's usual level of creative genius, but these tended to be somewhat contrived attempts to explain the master's *lapse*. However, no one of these studies considered the geometrical characteristics of the vaults, based on detailed surveys.

In some previous researches done by author, attempts were made to formulate new hypotheses on the form-finding processes that could have determined the shape of umbrella's vault sail. The analyses were done by correlating detailed digital surveys[1] and geometric analyses of the vault, comprising the curves, surfaces and the possible masonry texture with the hypothesized form.

For the complexity of argument that was treated in some previous papers (RADOJEVIC 2014, 2015), in this occasion we will give just a brief explanation of the design processes that led to the actual form, necessary to understand for the comprehension of this contribution.

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[1] The integrated survey (terrestrial laser scanner and photogrammetry), promoted by Opera di Santa Crocec was done in 2012 in a collaboration with Università di Firenze within the research project *"Laboratorio di Santa Croce"*. The survey team was composed by prof. Maria Teresa Bartoli and the author.

Geometry of the umbrella vault

The umbrella vault's is done on dodecagonal base, and the semi-circular stone ribs support the light double-sailed structure. The ribs are radiating from center of cupola where there is a circular lantern. The sails are double curved with a very complex geometry. Currently, we can see (and measure) only the inner sail's intrados; but we have some testimonials about the outer one. The survey designs, done by P. A. Rossi during the roof restauration period, show the whole structure, (fig. 3). From these documents we can see that the umbrella vault is double sailed and that the ribs are done by the radial brick layering (LASCHI, ROSSELLI and ROSSI 1962). The hypothesis made by the author considers the form of the inner sail that is obtained by a three-dimensional transformation of a toric surface with respect to the fixed point (dome's oculus) and constant length (torus radius); like the conchoid of Nicomedes in 2d.

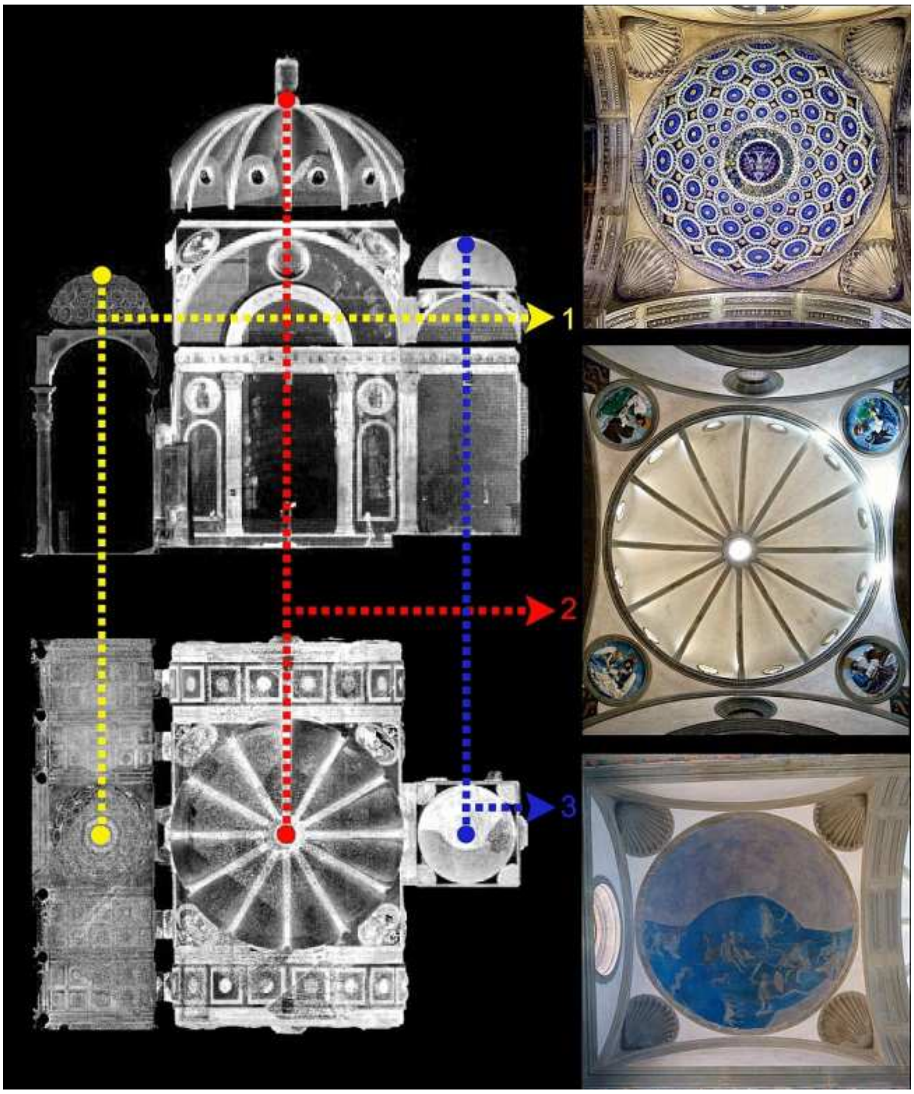

Fig. 2 – Point cloud snapshots that show the top orthographic view and the longitudinal section of Pazzi Chapel (on the left) and the three cupolas in this order (from top to bottom): the portico vault (related to the stereographic projection), the umbrella vault (torus solid projection) and the semispherical dome with astronomical fresco.

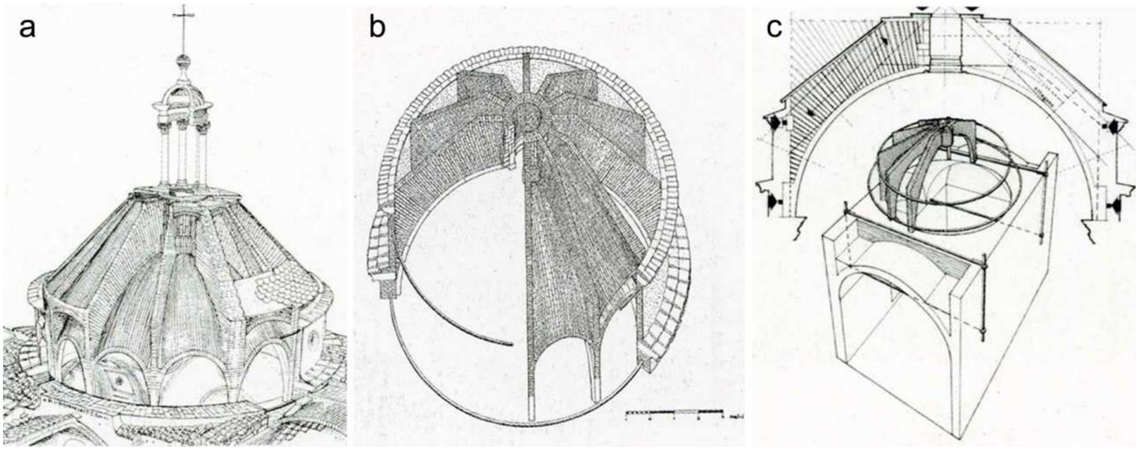

Fig. 3 – The sequence of images show the survey drawings done by arch. Paolo Alberto Rossi during the restauration works. From these documents we can see that the umbrella vault is double sailed and that the ribs are done by the radial brick layering. (Drawings by: P. A. Rossi)

a) Structural perspective drawing of the umbrella vault that shows the double sailed structure.

b) Axonometric drawing that shows a rib's structure.

c) Analytic structural diagram of the rib. (LASCHI, ROSSELLI and ROSSI 1962)

According to this hypothesis, the vault is made by the inner sail (the conchoid surface) and the outer sail that is currently not visible (the hypothesized toric surface). Thus obtained nurbs surface was overlapped by the mesh surface generated from the surveying data (the cloud of points). The models overlapped almost perfectly! Only 4% of points from the mesh model were out of the 3 cm range from the nurbs one (fig. 7). This very particular form, even though it explains the building process in a very satisfactory manner (RADOJEVIC 2015), is not admirable by the visitors of the Chapel[2]. Actually, the conchoid surface (that is a solid central projection of torus), could be perceived like a transformation of torus only under certain conditions. In this paper we will give a possible way to make these characteristics evident.

Design process

Premise

The geometrical survey based studies carried on previously by different scholars, were not successful in describing exact shape of the sail (BAGLIONI et al. 2007, SALEMI 2007). The descriptions given so far, despite drew near to geometry, did not give any explanation of the design process or of the constructive logic of such a form. The shape of the sail has been described as a form generated by series of circles that appertain to radial plans, with the variable radius. For the construction of that form infinite number of different centering would be needed. This way to construct that is not attune to Brunelleschi at all, who, just to remember, built the Dome's cupola without fixed centering. With these premises, we will try to describe a double sailed form that is possible to build without any centering (only with one cord).

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[2] The conchoid form, that is a three-dimensional transformation of a torus, is not perciaveble as such from the viewpoint that is different from the "transformation point", that is the oculus.

Another hypothesis, that would perfectly explain the construction issues, was the one of the toric surface; given by the two circular ribs that represent the Villarceau circles[3] (SALEMI 2007). The surface is generated by a constant radius circle (generatrix) that is moving along the circular path (directrix). The path is given by the torus equatorial circumference. Even though this hypothesis could perfectly explain the constructive issues thus obtained toric surface doesn't match with the surveying data; at least for the inner sail.
With these premises we will try to describe a new hypothesis for the double sailed dome's form; that should be both possible to build without many centering (in our case only one cord is needed) and to describe in a simple and logical manner.

Outer sail – Torus

If we try to think about a possible sail's form that has two circular directrices (given by the ribs) and a circular generatrix with a constant radius, the only solution that arises is the torus surface, where the vault ribs are the circumferences of Villarceau[4] (VILLARCEAU 1848, SCHMIDT 1950), as hypothesized by Salemi. Although, finding a torus from a pair of these circumferences is not easy when two symmetrical circles are given (the two ribs) the determination of torus suddenly becomes very simple (fig. 4). As the Villarceau circles represent also the rhumb lines of the torus, which can be simply obtained by placing each second brick crosswise, and their centers are distributed on a circumference that pertains to the vertical meridian plane it is theoretically possible to build a torus without any centering but only with one cord. The unique position of each brick that pertains to a given rhumb line is given by two conditions: it pertains to sphere given by the fixed length cord, the slope of the rhumb line is an integer and it corresponds to the brick's proportion (RADOJEVIC 2015).

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[3] In geometry, Villarceau circles are a pair of circles produced by cutting a torus obliquely through the center at a special angle. Given an arbitrary point on a torus, four circles can be drawn through it. One is in the plane (containing the point) parallel to the equatorial plane of the torus. Another is perpendicular to it. The other two are Villarceau circles. They are named after the French astronomer and mathematician Yvon Villarceau (1813–1883). In 1903 Mannheim showed that the Villarceau circles meet all of the parallel circular cross-sections of the torus at the same angle, so they represent the torus circular rhumb lines.

[4] For every point on the surface of a torus, we can trace exactly 4 distinct perfect circles, on the surface of the torus, that pass through that point: one is around the hole of the torus, and the other around its circumference. The other two are Villarceau circles, that appertain to the be-tangent planes.

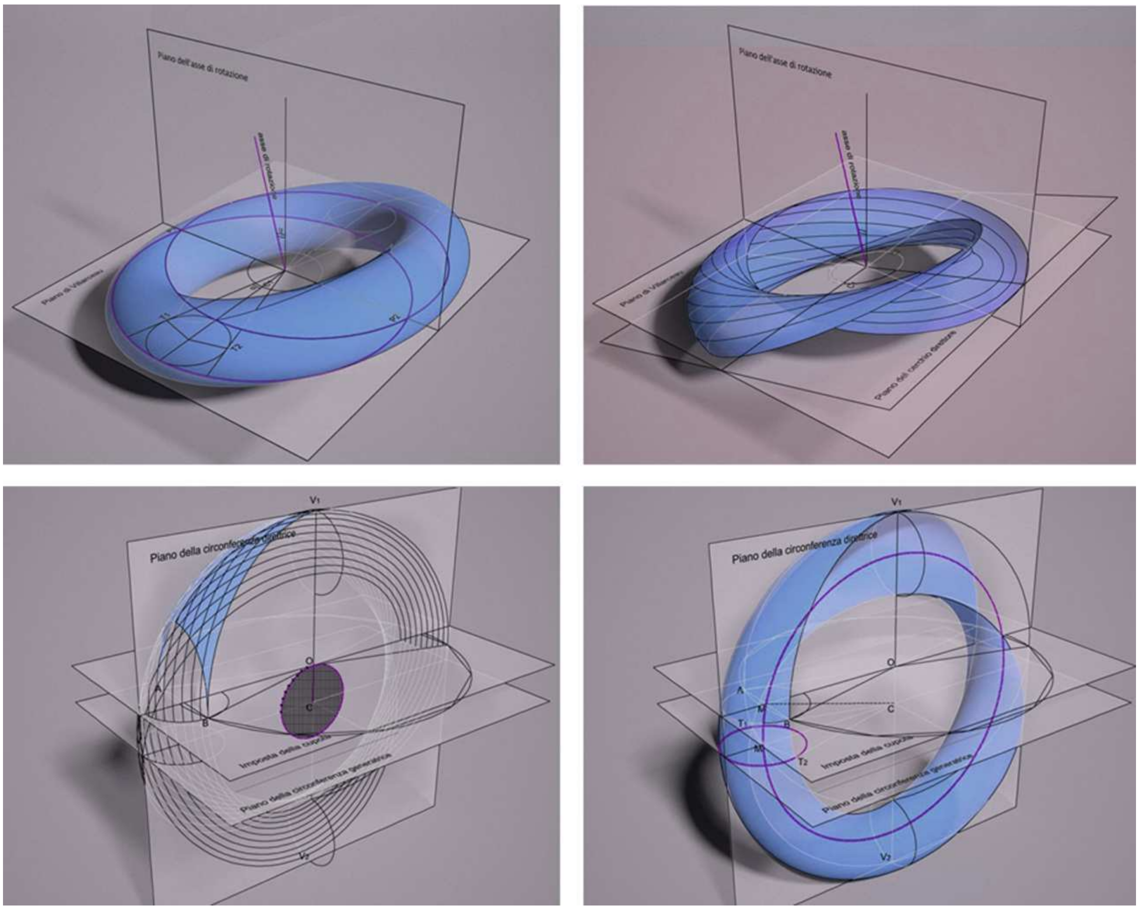

Fig. 4 – Construction of torus given by two Villarceau' circles. (Left, Below) The Pazzi Chapel's case: Given two ribs with the center in point, O, and radius, R, that represent the Villarceau circles we need to find the torus. The radius of the directrix circle of the torus is equal to the one of the given circle, R, and the radius of the generatrix circle of the torus, r, is given by the edge of the polygon (AB/2). The center of the torus, C, is on the vertical axis of the dome below cupolas center, C, for the distance r.

Inner sail – Conchoid of torus

The toric surface doesn't represent the inner sail in this case, as hypothesized by Salemi, but the outer one. Once that we have obtained this surface we can easily define also the inner sail's form. The inner sail is given by simple deformation of a given torus. The transformation of this three-dimensional surface follows the same principles used by Nicomedes to construct his famous conchoid in 2d (fig. 5). The conchoid of Nicomedes is a sort of a central projection of a line, and if it's seen from the focus point, O, it produces certain optical effects. Therefore it was used in determination of stem of entasis of Ionic columns.

The Conchoid

For every line through point O that intersects the given curve, c, at point M the two points on the line which are at the fixed distance, s, from the point M are on the conchoid. They are called conchoids because the shape of their outer branches resembles conch shells. Therefore, the conchoid is the locus of points P fixed distance, s, away from a curve, c, as measured along a line from the focus point, O (fig. 5).
In the Pazzi Chapel's case the focus point is the oculus, the given *curve*, which is a surface in this case, is torus and the fixed distance is a torus radius, r (fig. 6).

Thus obtained surface has been verified by overlapping the hypothesized NURBS model (conchoid sails) and the MESH model obtained from the cloud of points. The result was very surprising because the two surfaces overlapped almost perfectly (Figs. 7, 8).

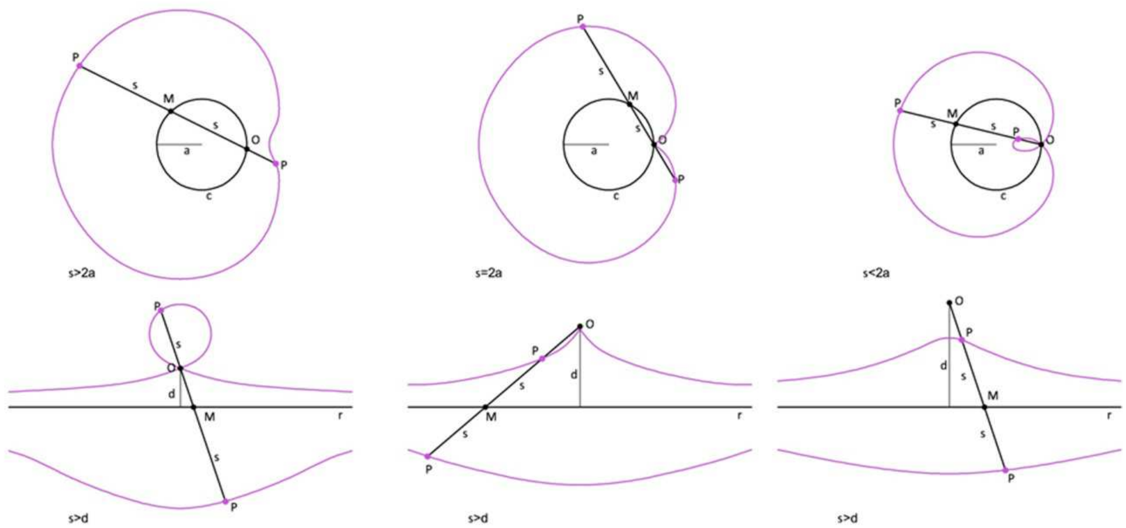

Fig. 5 – (Above) For every line through point O that intersects the given curve, c, at point M the two points on the line which are at the distance, s, from point M are on the conchoid. They are called conchoids because the shape of their outer branches resembles conch shells. Therefore, the conchoid is the locus of points, P, fixed distance, s, away from a curve, c, as measured along a line from the focus point, O. (Below) Nicomedes conchoid. Nicomedes recognized the three distinct forms seen in this family that depends on a relation between s and d as shown in the picture.

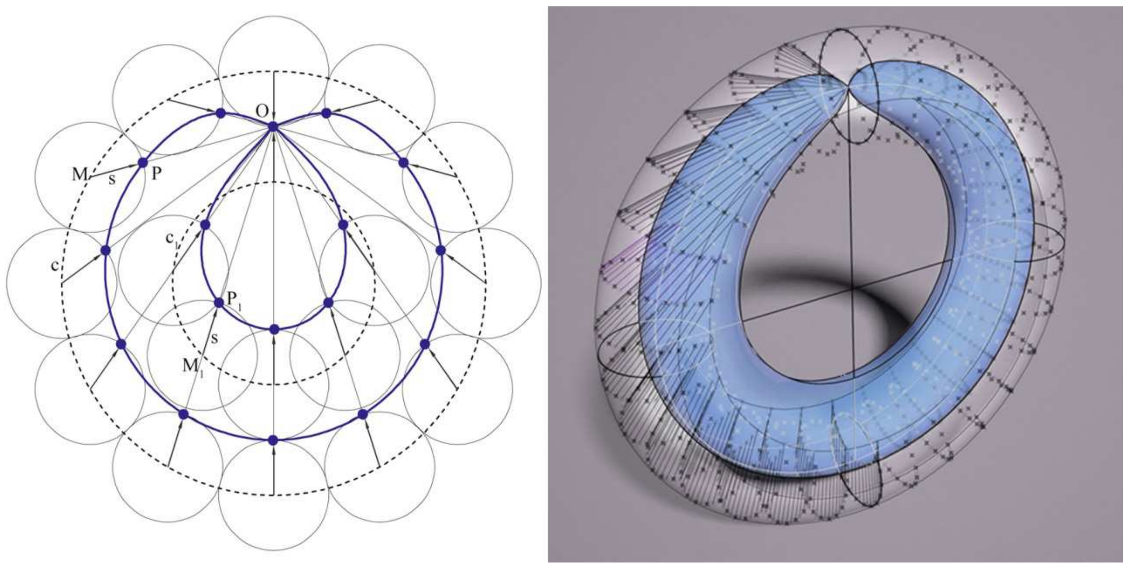

Fig. 6 – The conchoid of torus. (Left) Section. The dashed line represents a torus section. From the fixed point, O, that is cupola's oculus, we transform a circle, c, into a conchoid. The fixed length, s, is equal to torus minor radius, r. (Right) Perspective view.
The conchoid can be easily also be obtained by a point on circle (epicycle) that flows along another circle (deferent) similar to the planetary orbits pre-Copernican explanations.

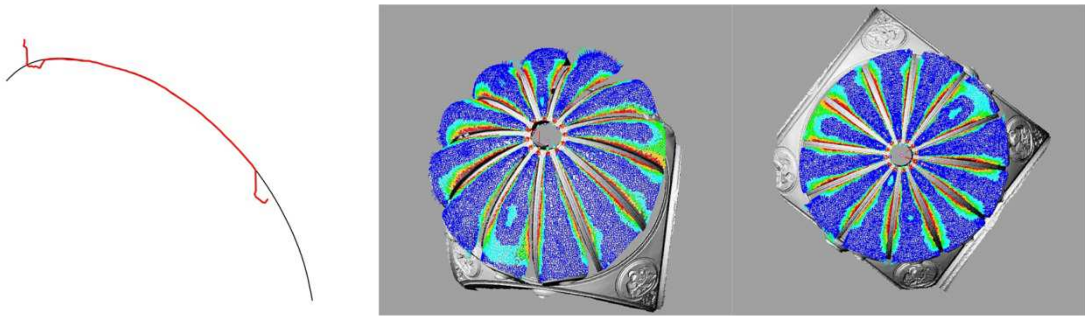

Fig. 7 – Verification of the hypothesized form. (Left) The sail section: the red line represents the MESH model section and the black line is the conchoid section. (Right) Tree-dimensional overlap. The colors represent distances between two models: Blue 0-3cm, Cyan 3-5 cm, Green 5-7 cm, Yellow 7-9 cm, Red more than 9 or missing zones

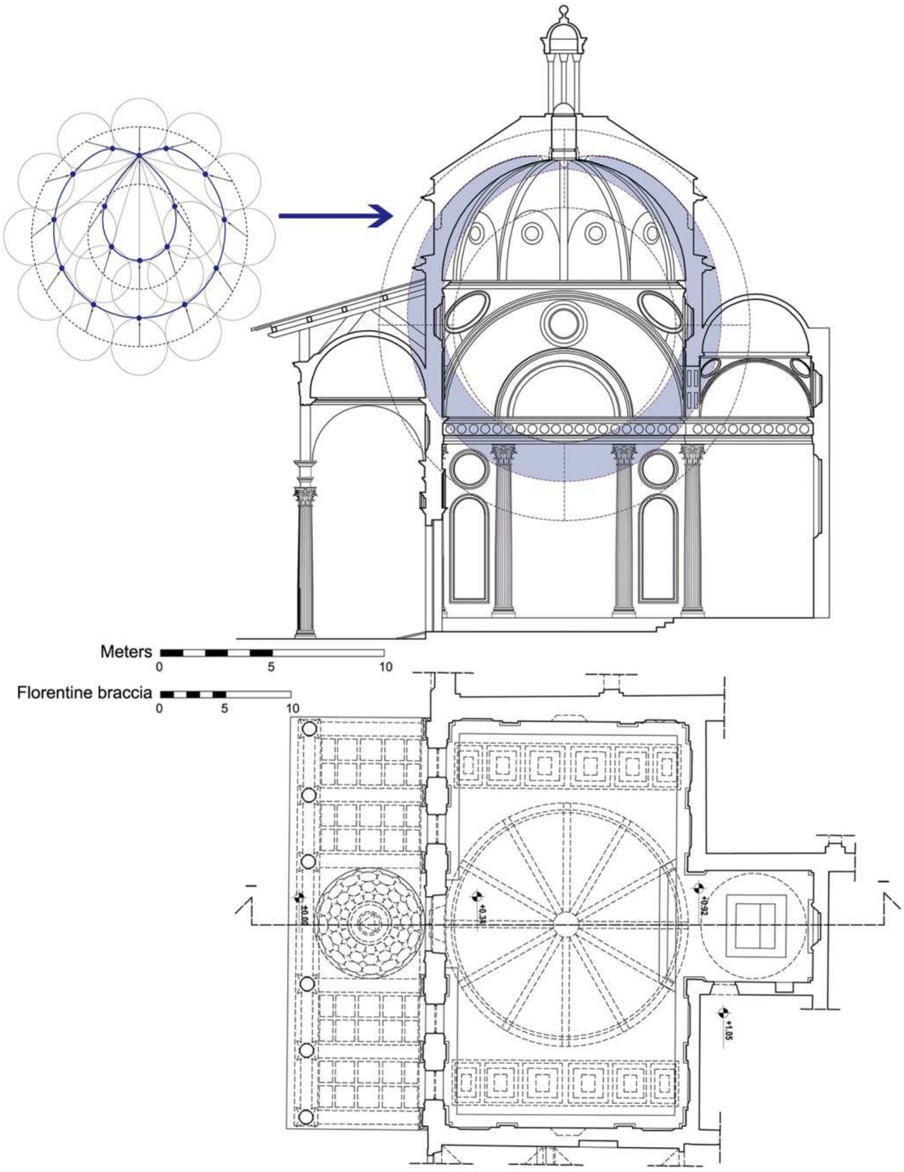

Fig. 8 – Plan and section. The survey is based on the cloud of points. The conchoid surface matches perfectly with the inner sail, but also the outer one (torus) is also possible because it fits under the roof.

Inspiration

All of the three cupolas of the Pazzi Chapel are in straight relation to the astronomical issues of the period (fig. 2) and, as we know, Brunelleschi was working together with the famous Renaissance astronomer Paolo del Pozzo Toscanelli on this project. In that period, the Ptolemaic system of a geocentric cosmology was entering the profound crisis and the explanation of the retrograde planet motions was becoming always more complex. With the center on Earth the apparent planetary motions were described as a complex set of movements given by a point on a circle that rotates both around its center (epicycle) and another circle (deferent). The number of circles, in some theories, was even more than two. One of the arguments used by Copernicus, as a proof of the heliocentric system, was that the visual center is wrong (it has to be moved to the sun) and the complex movement is only apparent (figs. 9, 10, 11). By moving the observer from the Earth onto the Sun the orbits return simply and circular.

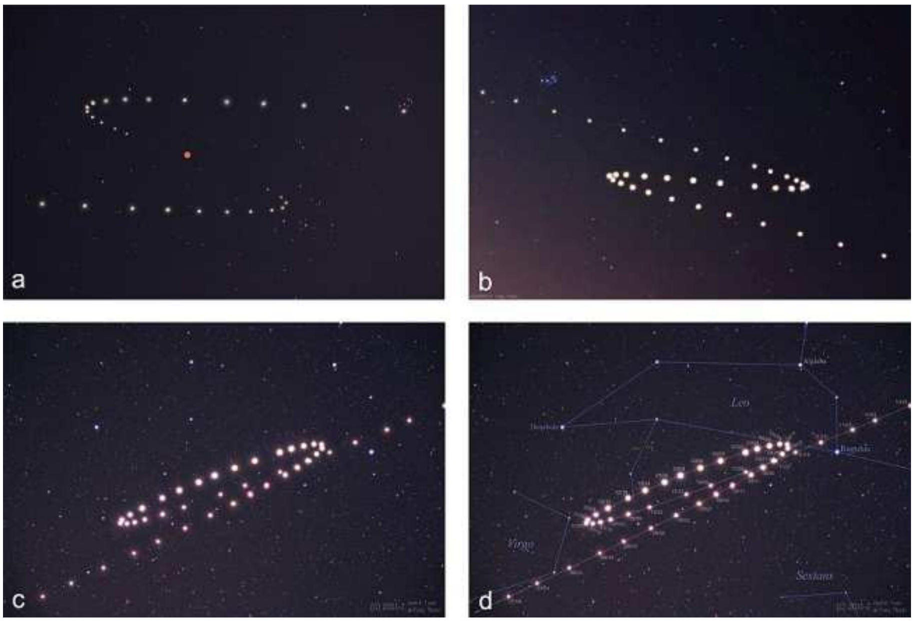

Fig. 9 – This sequence of images show the apparent retrograde motion of Venus (inner planet) and Mars (outer planet) seen from the Earth. (Credits and Copyrights: Tunc Tezel, TWAN organization, www.twanight.org)

a) Planet Venus traced out this S shape in Earth's sky during 2004. Following the second planet from the Sun in a series of 29 images recorded from April 3rd through August 7th (top right to bottom left) of that year, astronomer Tunc Tezel constructed this composite illustrating the wandering planet's path against the background stars. The series reveals Venus' apparent retrograde motion transporting it from a brilliant evening star to morning's celestial beacon.

b) This composite of images spaced about a week apart - from late July 2005 (bottom right) through February 2006 (top left) - traces the retrograde motion of Mars through planet Earth's night sky. On November 7th, 2005 the Red Planet was opposite the Sun in Earth's sky (at opposition). That date occurred at the center of this series with Mars near its closest and brightest. But Mars didn't actually reverse the direction of its orbit to trace out the Z-shape. Instead, the apparent backwards or retrograde motion with respect to the background stars is a reflection of the motion of the Earth itself. Retrograde motion can be seen each time Earth overtakes and laps planets orbiting farther from the Sun, the Earth moving more rapidly through its own relatively close-in orbit. The familiar Pleiades star cluster lies at the upper left.

c, d) This composite of images spaced some 5 to 7 days apart from late October 2011 (top right) through early July 2012 (bottom left), traces the retrograde motion of Mars through planet Earth's night sky. On March 4th, 2012 Mars was opposite the Sun in Earth's sky, near its closest and brightest at the center of this picture.

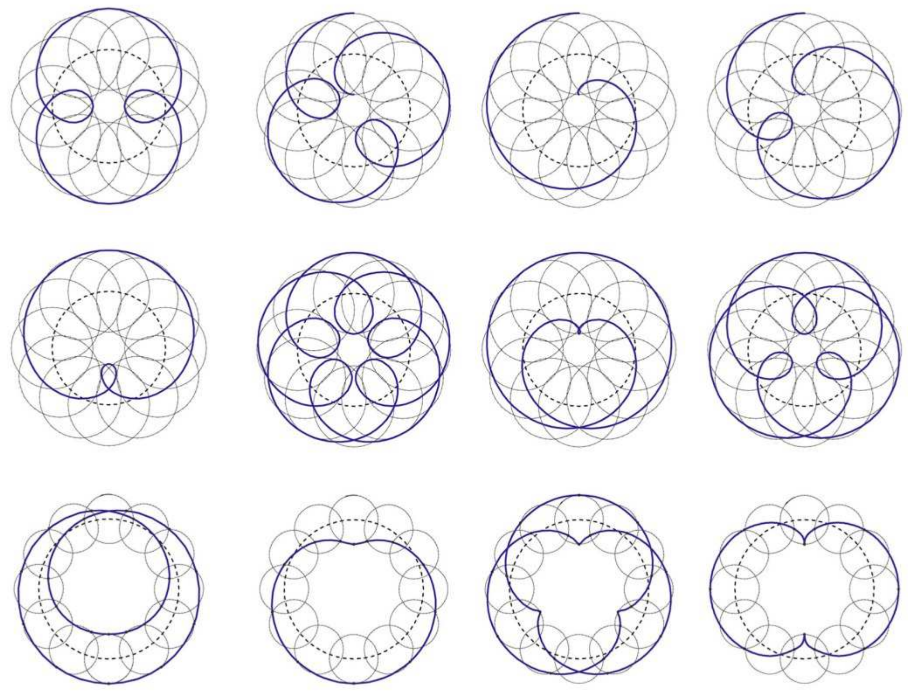

Fig. 10 – Epicycles and deferents in different relations. Pre-Copernican explanation of the irregular planetary orbits. The Ptolemaic model accounted for the apparent motions of the planets in a very direct way, by assuming that each planet moved on a small sphere or circle, called an epicycle, that moved on a larger sphere or circle, called a deferent. Some of the designs (like second in the third line) are very similar to the Pazzi Chapel's section.

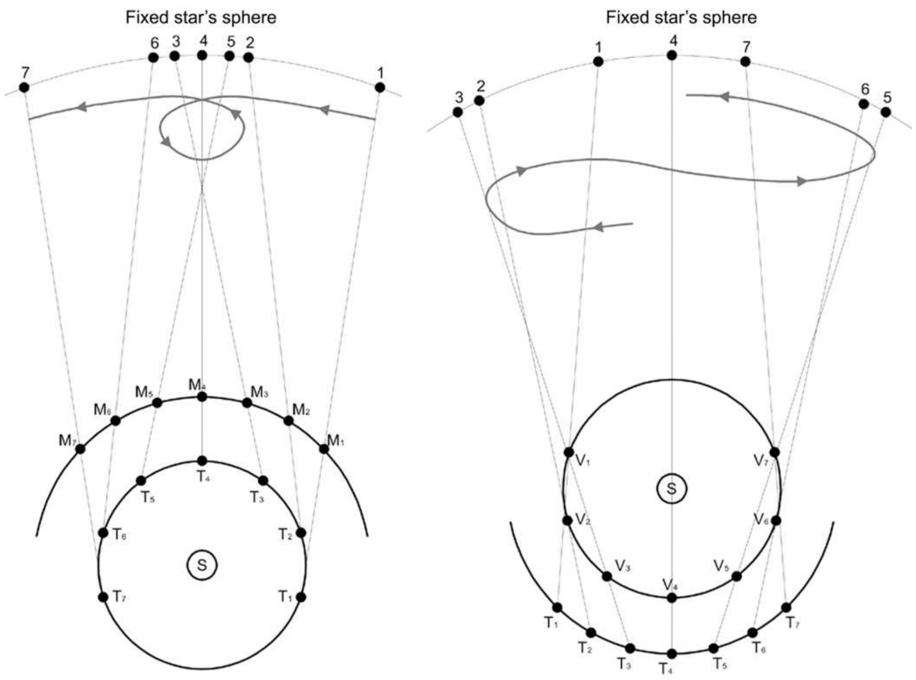

Fig. 11 – Copernicus explanation of a retrograde motion. (Left) The outer planet seen from the Earth, that is not center of the universe. The apparent retrograde motion of Mars is given by the different velocities of revolution around the sun between two planets (Earth and Mars). (Right) The inner planet, Venus, that is orbiting around the Sun faster than Earth.

The Copernicus solution of the retrograde motion issue is based on the same principles (central projection ones) that led Brunelleschi to discover the linear perspective. Therefore he surly could understand very well these issues and might have also discussed them with Toscanelli. The conchoid form can be perfectly imagined as a circle that rotates along another circle (as epicycle and deferent) or, in three-dimensional space, like a sphere that flows along the toric surface. The center of the sphere appertains to the toric surface and its radius is equal to the torus minor one, r.

Why should we look the umbrella vault from its oculus?

The conchoid form, as we mentioned at the beginning, is also a sort of a solid central projection of torus, with the fixed point (view point) in its oculus. That means that if seen from that point it shouldn't appear different from the torus. Two questions arise: How are we going to be sure that the form we see is a torus (or its projection)? How are we going to move the observer's view point to the oculus?

Mapping the torus

The peculiar characteristics of toric surface are its circles: meridian, parallel and the radial ones. If we could find a sort of a torus stereographic projection[5], where those circles remain circles (as if it happens for the sphere) we could use it for the demonstration, but there is no such a projection in three dimensional space[6]. Anyway, if we try to find the positions of the plans, on which the projections of torus radial circles from oculus remain circles, surprisingly we arrive very near to the solution. The solution is not exact, but the planes for different circles are almost parallel; the maximum angle between them is only 2º, and the deformation of a circles is not perceivable (the eccentricity is similar to the terrestrial orbit's one)! For the other viewpoints this angle is much greater.

[5] In geometry, the stereographic projection is a particular mapping (function) that projects a sphere onto a plane. The projection point is the pole and the projection plane is an equator. The projection is defined on the entire sphere, except at one point: the projection point. Where it is defined, the mapping is smooth and bijective. It is conformal, meaning that it preserves angles. It is neither isometric nor area-preserving: that is, it preserves neither distances nor the areas of figures.

[6] The stereographic projection of torus is known to the mathematicians but only in a four dimensional space. In geometric topology, the Clifford torus is a special kind of torus sitting inside the unit 3-sphere S3 in R4, the Euclidean space of four dimensions. Or equivalently, it can be seen as a torus sitting inside C2 since C2 is topologically equivalent to R4. It is specifically the torus in S3 that is geometrically the cartesian product of two circles, each of radius sqrt(1/2).

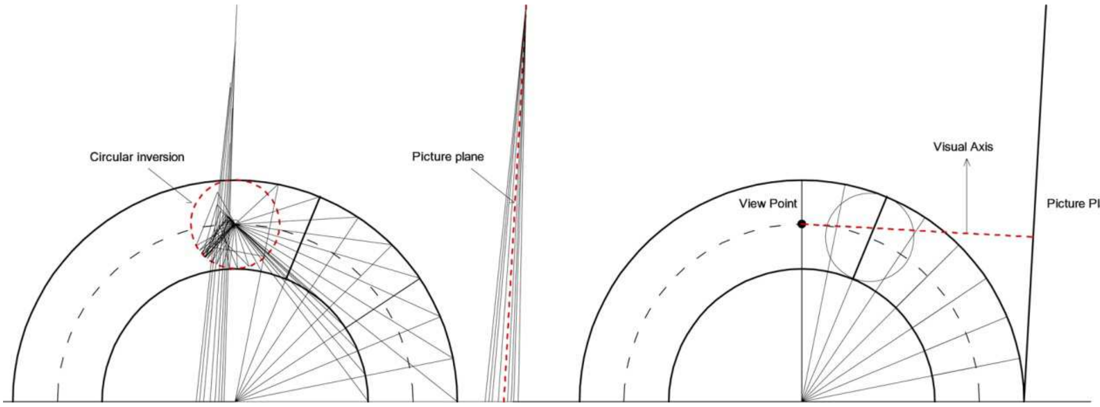

Fig. 12 – Determination of the picture plane for the shown series of radial circumferences by the use of the circular inversion. For the given viewpoint the planes where these circumferences transform conformably are almost parallel, as shown in the picture. In the figure 14 we can see the transformation of series of radial circles onto the given plan.

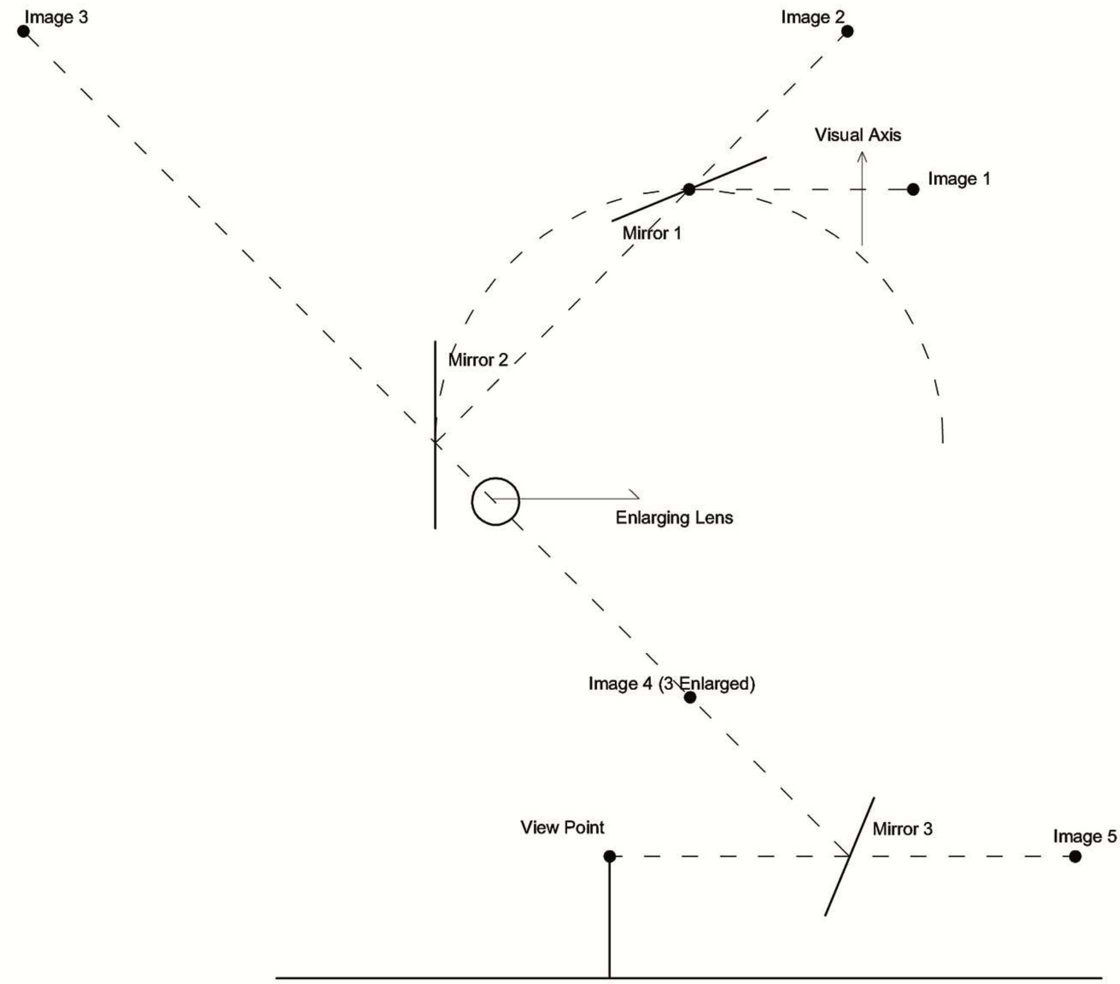

Fig. 13 – The system of mirrors that is "taking" the observer's viewpoint to the oculus

At this point, we need to map the conchoid surface with the central projection (from oculus) of these circles onto its surface. These curves onto the conchoid surface are not circular any more, but seen from oculus (projection center) they will appear as circles.

Moving the observer's viewpoint to the oculus

Once that we mapped the inner sail with the described curves we need to move the observers viewpoint to the oculus. For that purpose we can use a system of mirrors and enlargement lens as shown in a figure (fig. 13). The observer that stays on the ground can see two very different images of the sail. If he is looking up to the sail he sees the complex curves and if he is looking at the mirror he sees concentric circles (fig. 14).

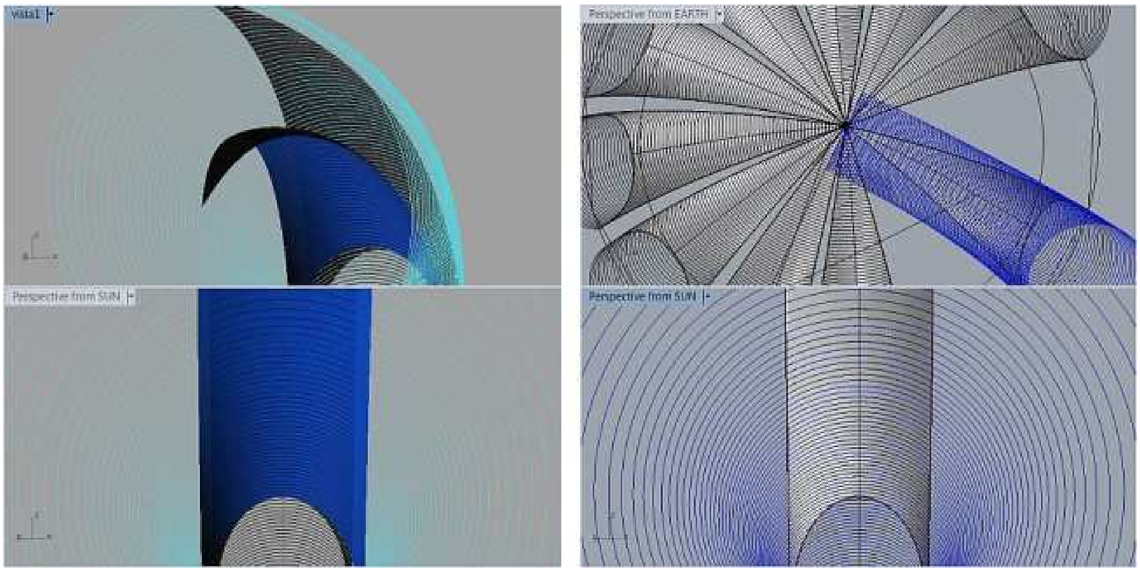

Fig. 14 – The image that the observer sees from the ground (above) and the image that is seen in the mirror (below), that is reflecting the oculus' viewpoint. The lines that are projected onto the conchoid surface, that represent the centr al projection of the torus radial circumferences, become circular in the perspective from oculus. The view point and the perspective plan are shown in the figure 12.

Conclusions

This, very particular and brilliant design choice, might have been inspired by the scientific taught of the period. In that particular moment, the Ptolemaic model of the Universe was showing its weakness, and the new proposals, which would resolve some issues, speculate the planetary motions, which generate new and dynamic figures. The fact that the apparently complex motions of planets could be actually very simple if observed from another point could be easily intuited by Brunelleschi. The same sets of rules that generate the pictures in central projection were designing the orbits in our sky. The shape of the sail, which to an observer on the floor (Earth) shows all its complexity, seen from the oculus (Sun) reveals its simple geometry, that is not unlike a torus.

At this point it is difficult to think that somebody else apart from Brunelleschi could have done this very particular project, that form is perfect also from the constructive point of view. Even if he taught of it for Old Sacristy it was obviously not easy, or almost impossible, for someone else to copy.

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